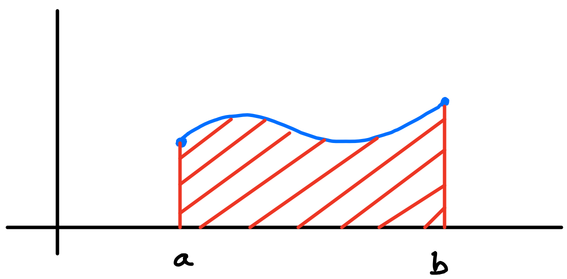
The Definite Integral

f - continuous function on $[a,b]$. $f(x) \geq 0$ for all x in $[a,b]$.



= Region enclosed by $y = f(x)$ and x-axis, between a and b.

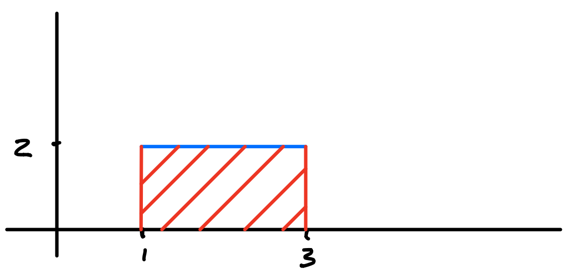
"

Area "under" curve between a and b

Q: What is the area of ?

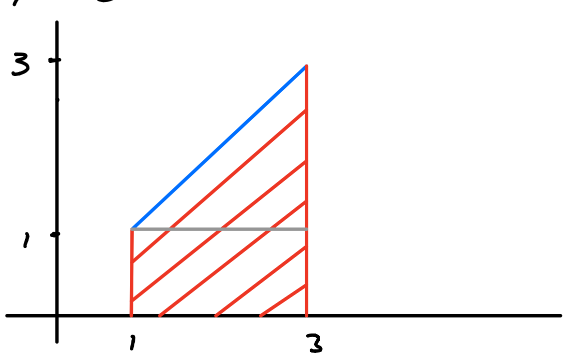
Example

1) $y = 2$, $a = 1$, $b = 3$



$\Rightarrow$ Area ( ) $=$ Base $\times$ Height $= 2 \times 2 = 4$

2) $y = x$, $a = 1$, $b = 3$

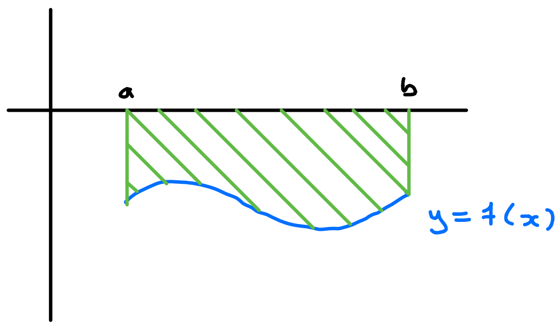


$\Rightarrow$ Area ( ) $=$ Area of triangle $+$ Area of rectangle

$$= \frac{2 \times 2}{2} + 2 \times 1$$

$$= 2 + 2 = 4$$

Assume now $f(x) \leq 0$ on $[a,b]$.

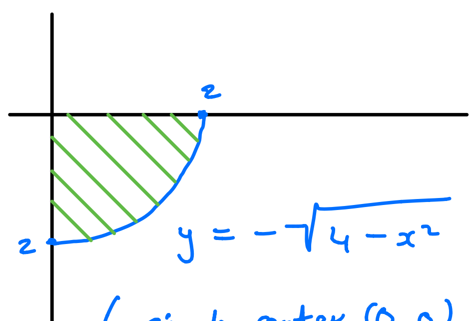


= Region enclosed by $y = f(x)$ and x-axis, between a and b.

" Area "over" area between a and b

Q: What is Area( ) ?

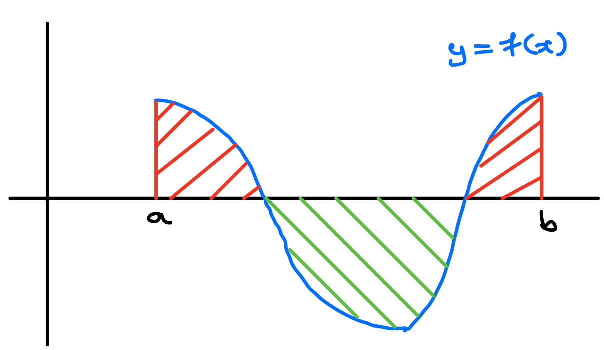
<u>Example</u> : $y = -\sqrt{4-x^2}$, $a = 0$, $b = 2$



(circle center (0,0) radius 2)

$\Rightarrow$ Area( ) $= \dfrac{\pi \cdot 2^2}{4} = \pi$

More generally : Do not assume f always positive or negative.



= Region enclosed by $y=f(x)$ between a and b <u>above</u> x-axis

= Region enclosed by $y=f(x)$ between a and b <u>below</u> x-axis

Definition

<u>Total Area</u> enclosed by $y = f(x)$ and x-axis between a and b $= \text{Area}(///) + \text{Area}(\backslash\backslash\backslash)$

<u>Net Area</u> enclosed by $y = f(x)$ and x-axis between a and b $= \text{Area}(///) - \text{Area}(\backslash\backslash\backslash)$

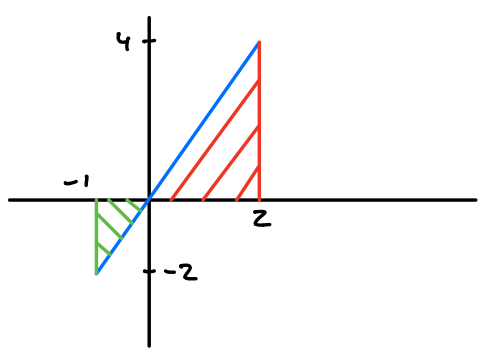
Called the "definite integral" of $f(x)$ on $[a,b]$

$$\int_a^b f(x)\,dx = \underline{\text{Net Area}} = \text{Area}(///) - \text{Area}(\backslash\backslash\backslash)$$

<u>Warning</u> : Definite Integral $\neq$ Indefinite Integral

Just a number — A general antiderivative

<u>Example</u> $\displaystyle\int_{-1}^{2} 2x\,dx = ?$



$$\text{Area}(///) = \frac{2 \times 4}{2} = 4$$

$$\Rightarrow \text{Area}(\backslash\backslash\backslash) = \frac{1 \times 2}{2} = 1$$

$$\Rightarrow \int_{-1}^{2} 2x\,dx = 4 - 1 = 3$$

Calculus

Derivatives

Intuitive Definition : Slopes of tangent lines

Precise Definition : $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Definite Integrals

Intuitive Definition : Net Area

Precise Definition : ?

English definition!

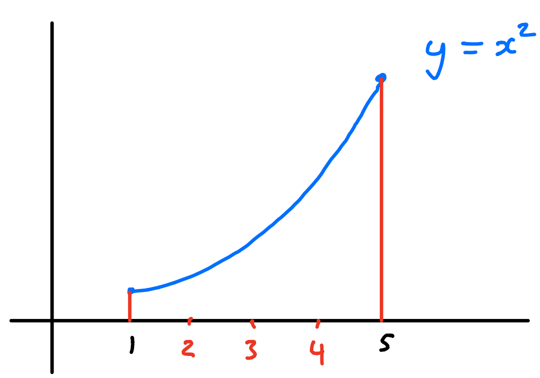
Integrate : "To combine many parts to make a whole".

Strategy : Approximate a region using rectangles of equal width.

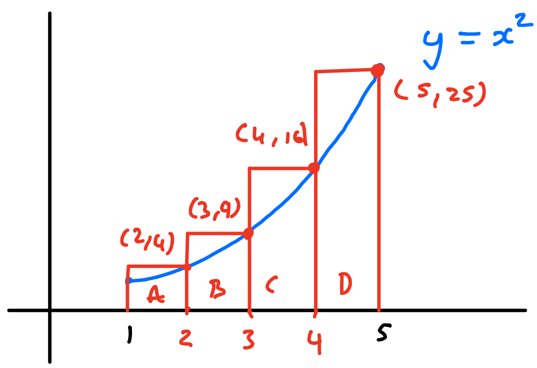
Example $f(x) = x^2$, $a = 1$, $b = 5$, $[1, 5]$

Step 1 Choose number of rectangles $n = 4$.

$\Rightarrow$ Width of each is $\frac{5-1}{4} = 1$. Divide $[1,5]$ into 4 intervals each of length 1.



Step 2   Choose heights of each rectangle by evaluating x^2 at end right endpoints of each interval.



Step 3   Calculate each area and add.

$$\left.\begin{aligned} \text{Area}(A) &= 1 \times 4 = 4 \\ \text{Area}(B) &= 1 \times 9 = 9 \\ \text{Area}(C) &= 1 \times 16 = 16 \\ \text{Area}(D) &= 1 \times 25 = 25 \end{aligned}\right\} \Rightarrow \text{Area under } y = x^2 \text{ between } 1 \text{ and } 5 \approx \underset{\displaystyle 51}{\underset{\shortparallel}{4+9+16+25}}$$

Problem : Not a good approximation as the rectangles are too wide.

Clever Idea : Increase number of rectangles making them narrower and narrower.

General Situation : $f(x) \geq 0$ on $[a, b]$.

<u>Step 1</u> Choose a number of rectangles n.

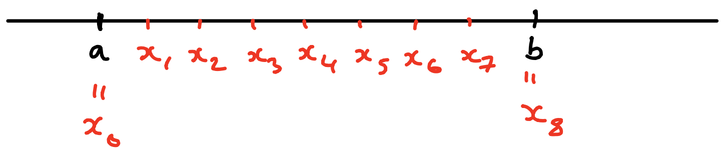
Divide $[a, b]$ into n equal intervals, each

just notation

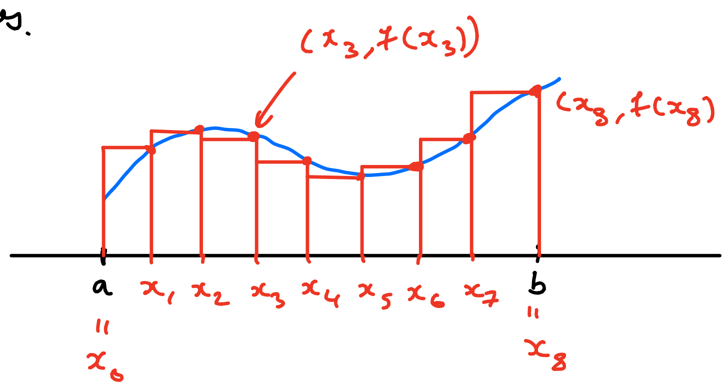
with length $\dfrac{b-a}{n} = \Delta x$. Each rectangle

will have width Δx.

E.g. $n = 8$



<u>Step2</u> Evaluate f at the right endpoints of each subinterval. These will be the heights of the rectangles.



<u>Step3</u> Calculate each area and sum

$$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$\Rightarrow$ Area under curve $y = f(x)$ between a and b $\approx f(x_1)\Delta x + \dots + f(x_n)\Delta x$

Better the bigger n is

Called Riemann Sum

Precise Definition of Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(f(x_1)\Delta x + \dots + f(x_n)\Delta x \right)$$

Remark

1/ Could use any points in the sub intervals to get heights (e.g. left endpoint, midpoints). The end result would be the same.

2/ When $f(x) \leq 0$ the rectangle area is counted as a negative. Thats why

$$\int_a^b f(x)\,dx = \underline{\text{Net Area}}$$

3/ This may be the precise definition but it isn't useful for calculation. Riemann Sums are necessary but cumbersome.